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ON DS FUNCTOR FOR AFFINE LIE SUPERALGEBRAS

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ABSTRACT. We study Duflo-Serganova functor for non-twisted affine Lie superalgebras and affine vertex superalgebras.

0. INTRODUCTION

Let $\mathfrak{g}$ be a Lie superalgebra and let $x \in \mathfrak{g}_{\bar{1}}$ satisfy the condition $[x,x] = 0$. The operator ad_x defines an odd square zero endomorphism of any $\mathfrak{g}$-module. This yields a functor $N \mapsto \mathrm{DS}_x(N) := Ker_N x/Im_N x$ from the category of $\mathfrak{g}$-modules to the category of modules over $\mathfrak{g}_x := \mathrm{DS}_x(\mathfrak{g})$.

The functor DS_x was introduced in [DS] (see also [S2]) as a means to assign an analog of singular support to representations of Lie superalgebras. This functor preserves superdimension and tensor product of representations.

Recall that the *defect* of a finite-dimensional Lie superalgebra $\mathfrak{g}$ is the dimension of a maximal isotropic subspace in $\mathbb{Q}\Delta$; for $A(m-1,n-1), B(m,n), D(m,n)$ the defect is equal to $min(m,n)$; for other cases of non Lie algebras it is one. It is well-known that the defect is equal to the maximal number of mutually orthogonal isotropic simple roots. A finite-dimensional simple Lie superalgebra of zero defect is either a simple Lie algebra or $\mathfrak{osp}(1|2l)$; the finite-dimensional modules over these Lie superalgebras are completely reducible (and these are the only simple Lie superalgebras with this property).

If $\mathfrak{g}$ is a finite-dimensional Lie superalgebra, then $\mathrm{DS}_x(\mathfrak{g})$ is a finite-dimensional Lie superalgebra of a smaller defect. If $\mathfrak{g}$ is the affinization of $\dot{\mathfrak{g}}$ and $x \in \dot{\mathfrak{g}}_{\bar{1}}$, then $\mathfrak{g}_x$ is the affinization of $\dot{\mathfrak{g}}_x$, see [GS].

In this paper we consider the DS functors for affine Lie superalgebras $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ and affine vertex superalgebras $V_k(\mathfrak{g})$; we always assume that $x \in \dot{\mathfrak{g}}_{\bar{1}}$.

Let $Vac^k(\mathfrak{g})$ be a vacuum $\mathfrak{g}$-module of level k and $Vac_k(\mathfrak{g})$ be its simple quotient. It is easy to see that $\mathrm{DS}_x(Vac^k(\mathfrak{g})) = Vac^k(\mathrm{DS}_x(\mathfrak{g}))$. We prove that for a non-negative integral k one has $\mathrm{DS}_x(Vac_k(\mathfrak{g})) = Vac_k(\mathrm{DS}_x(\mathfrak{g}))$ if $\dot{\mathfrak{g}}_x$ has zero defect and $\dot{\mathfrak{g}}_x \neq \mathbb{C}$, see Theorem 2.2. As a result, the corresponding vertex algebras are isomorphic, see Corollary 3.4.2.

The principal admissible modules for an affine Lie algebra $\mathfrak{t}$ were classified in [KW5]. A level k is called principal admissible if $Vac_k(\mathfrak{t})$ is principal admissible. From Theorem

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of Arakawa [A] it follows that for a principal admissible level k the $V_k(\mathfrak{t})$-modules in the category $\mathcal{O}$ are completely reducible and the irreducible modules are the principal admissible modules of level k.

We introduce the principal admissible levels for an affine Lie superalgebra $\mathfrak{g}$ using Kac-Wakimoto definition for Lie algebra case. We prove that if $\dot{\mathfrak{g}}_x$ is a simple Lie algebra and $\dot{\mathfrak{g}} \neq B(n+1|n)$, then for a principal admissible level k one has $\mathrm{DS}_x(Vac_k(\mathfrak{g})) = Vac_k(\mathfrak{g}_x)$. This implies the isomorphism of the corresponding vertex algebras. The proof is based on Arakawa's Theorem and the fact that the maximal proper submodule in $Vac^k(\mathfrak{g}_x)$ is generated by a singular vector (if $\mathfrak{g}_x$ is a Lie algebra, this can be easily deduced from [F]). We believe that the statement holds for $\mathfrak{g}_x = \mathfrak{osp}(1|2n)^{(1)}$, however both Arakawa's and Fiebig's results are not established in this case.

In Section 1 we recall the construction of Duflo-Serganova functor DS_x and summarize the results which we use later.

In Section 2 we study DS functor for integrable vacuum modules and prove Theorem 2.2. Since integrable vacuum modules have principal admissible levels, this theorem for the case, when $\dot{\mathfrak{g}}_x$ is a simple Lie algebra and $\dot{\mathfrak{g}} \neq B(n+1|n)$, is a particular case of Theorem 4.4.2. However, the proof of Theorem 2.2 is different: it does not use vertex algebras and Arakawa's Theorem. In § 2.4 we give an example when $\mathrm{DS}_x(Vac_k(\mathfrak{g})) \neq Vac_k(\mathfrak{g}_x)$ ($\mathfrak{g} = \mathfrak{sl}(1|2)^{(1)}$, k is critical).

In Section 3 we introduce the DS functor for vertex superalgebras. In particular, we prove that if $\mathrm{DS}_x(Vac_k(\mathfrak{g})) = Vac_k(\mathfrak{g}_x)$, then DS_x maps the simple affine vertex superalgebra $V_k(\mathfrak{g})$ to the simple affine vertex superalgebra $V_k(\mathfrak{g}_x)$. As a result, for any $V_k(\mathfrak{g})$-module N the image $\mathrm{DS}_x(N)$ is a $V_k(\mathfrak{g}_x)$-module.

In Section 4 we study $Vac_k(\mathfrak{g})$ if k is a principal admissible level, (this notion we define in § 4.2 similarly to the Lie algebra case). In § 4.4 we prove that $\mathrm{DS}_x(Vac_k(\mathfrak{g})) = Vac_k(\mathfrak{g}_x)$ if $\dot{\mathfrak{g}}_x$ is a simple Lie algebra and $\dot{\mathfrak{g}} \neq B(n+1|n)$.

Let $\dot{\Sigma}$ be a set of simple roots which contains a maximal isotropic subset $S = \{\beta_1, \dots, \beta_r\}$ (r is the defect of $\dot{\mathfrak{g}}$). We consider DS_x for $x = \sum_{i=1}^r x_i$, where x_i is a non-zero vector in $\mathfrak{g}_{\beta_i}$. In this case $\dot{\mathfrak{g}}_x = \mathrm{DS}_x(\dot{\mathfrak{g}})$ has zero defect. All our results are valid also for the composition $DS_S := DS_{x_1} \circ DS_{x_2} \circ \dots \circ DS_{x_r}$. Note that $DS_S(\mathfrak{g}) \cong \mathrm{DS}_x(\mathfrak{g})$.

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1. Preliminaries

Throughout the paper $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$, where $\dot{\mathfrak{g}} \neq D(2|1,a)$ is a finite-dimensional Kac-Moody superalgebra with a set of simple roots (a base) $\dot{\Sigma}$ and a Cartan subalgebra $\dot{\mathfrak{h}}$. We denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$: $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$.

Let $\Delta \subset \mathfrak{h}^*$ (rep., $\dot{\Delta}$) be the set of roots of $\mathfrak{g}$ (resp., of $\dot{\mathfrak{g}}$). We denote by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ the subsets of even and odd roots. We denote by W the Weyl group of $\mathfrak{g}_{\bar{0}}$. Recall that $\Delta_{\bar{0}}$ is a union of a finite number of root systems of affine Lie algebras with the same minimal imaginary root δ. Throughout the paper we fix $\dot{\Delta}$ and denote by Λ_0 the corresponding fundamental weight, i.e. $(\Lambda_0, \delta) = 1$ and $(\Lambda_0, \dot{\Delta}) = (\Lambda_0, \Lambda_0) = 0$.

We assume that $\dot{\Delta}$ is indecomposable (i.e., $\mathfrak{g}, \dot{\mathfrak{g}}$ are quasisimple in the sense of [S3]). We fix $\Delta^+_{\bar{0}}$ and consider the subsets of positive roots Δ^+ which contain $\Delta^+_{\bar{0}}$. The choice of Δ^+ gives a triangular decomposition of $\mathfrak{g}$, compatible with the triangular decomposition of $\mathfrak{g}_{\bar{0}}$, corresponding to $\Delta^+_{\bar{0}}$. For a fixed subset of positive roots Δ^+ we denote by Σ the corresponding base (i.e., the set of simple roots) and by ρ the corresponding Weyl vector. For a fixed base Σ we denote by α_0 the affine root, i.e. $\Sigma = \dot{\Sigma} \cup \{\alpha_0\}$, where $\dot{\Sigma}$ is the base of $\dot{\Delta}^+$.

We denote by $\mathcal{O}$ the BGG category of finitely generated $\mathfrak{g}$-modules with a diagonal action of $\mathfrak{h}$ and a locally finite action of $\mathfrak{g}_\alpha$ with $\alpha \in \Delta^+_{\bar{0}}$. The category $\mathcal{O}$ is equipped by a duality functor $N \mapsto N^\#$ and the simple modules are self-dual.

We normalize the form $(-,-)$ on $\mathfrak{g}$ as in [KW3] and set

$$\dot{\Delta}^\# := \{\alpha \in \dot{\Delta}_{\bar{0}} |\ (\alpha, \alpha) > 0\}.$$

The corresponding algebra $\dot{\mathfrak{g}}^\#$ is a simple Lie algebra; we denote its highest root by θ. We will use bases $\dot{\Sigma}$ such that θ is the highest root in $\Delta^+(\dot{\Sigma})$; then

$$\alpha_0 = \delta - \theta.$$

Let $\dot{\Omega}$ be the Casimir operator for $\dot{\mathfrak{g}}$ which corresponds to the invariant bilinear form $(-,-)$, see [K1], Ch. II. Recall that the dual Coxeter number $h^\vee$ is half of the eigenvalue of the Casimir operator $\dot{\Omega}$ on the adjoint representation $\dot{\mathfrak{g}}$ and $h^\vee = (\rho, \delta)$. We always choose the Weyl vector ρ in the form $\rho = h^\vee \Lambda_0 + \dot{\rho}$.

We say that $k \in \mathbb{C}$ is non-critical if $k \neq -h^\vee$ and $\lambda \in \mathfrak{h}^*$ is non-critical if $K(\lambda) \neq -h^\vee$, i.e. $(\lambda + \rho, \delta) \neq 0$.

We use the following notations: if $X, Y \subset \mathfrak{h}^*$ we set $(X, Y) = \{(x, y) |\ x \in X, y \in Y\}$; for a vector space V, $X \subset V$ and $R \subset \mathbb{C}$ we use the notation $RX = \{\sum_{i=1}^s r_i x_i |\ r_i \in R, x_i \in X\}$ (for instance, $\mathbb{Z}\Delta$ is the root lattice). For $S \subset \mathfrak{h}^*$ we set

$$S^\perp := \{\nu \in \mathfrak{h}^* |\ \forall \beta \in S\ (\beta, \nu) = 0\}.$$

If $\alpha \in \Delta$ is a non-isotropic root, we say that a $\mathfrak{g}$-module N is α-integrable if $\mathfrak{g}_{\pm\alpha}$ act locally nilpotently on N. We use conventions of [GK]. We say that a $\mathfrak{g}$-module $N \in \mathcal{O}$ is *integrable* if N is integrable as a $\dot{\mathfrak{g}}_{\bar{0}}$-module and N is α-integrable for each $\alpha \in \Delta$ satisfying $||\alpha||^2 > 0$.

For $A(m|1)^{(1)}, C(m)^{(1)}$, $N \in \mathcal{O}$ is integrable if N is integrable as a $\mathfrak{g}_{\bar{0}}$-module and $\mathfrak{h}$ acts diagonally.

1.1. **DS functor for affine Lie superalgebras.** Take $x \in \mathfrak{g}_{\bar{1}}$ satisfying $[x, x] = 0$. Recall that Duflo-Serganova functor DS_x is defined by

$$\mathrm{DS}_x(N) := Ker_N x / Im_N x;$$

we view $\mathrm{DS}_x(N)$ as a module over $\mathfrak{g}^x$ (where $\mathfrak{g}^x$ is the centralizer of x in $\mathfrak{g}$). Note that $[x, \mathfrak{g}] \subset \mathfrak{g}^x$ acts trivially on $\mathrm{DS}_x(N)$ and that $\mathfrak{g}_x := \mathrm{DS}_x(\mathfrak{g}) = \mathfrak{g}^x/[x, \mathfrak{g}]$ is a Lie superalgebra. Thus $\mathrm{DS}_x(N)$ is a $\mathfrak{g}_x$-module and DS_x is a functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules. This is a tensor functor ($\mathrm{DS}_x(N \otimes N') = \mathrm{DS}_x(N) \otimes \mathrm{DS}_x(N')$, see [DS]).

An exact sequence of $\mathfrak{g}$-modules

$$0 \to N_1 \to N \to N_2 \to 0$$

induces the exact sequence of $\mathfrak{g}_x$-modules

$$0 \to E \to \mathrm{DS}_x(N_1) \to \mathrm{DS}_x(N) \to \mathrm{DS}_x(N_2) \to \Pi(E) \to 0.$$

Recall that for a $\mathfrak{g}$-module N with a diagonal action of $\mathfrak{h}$ one has

$$\mathrm{sch}\, N := \sum_{\nu \in \mathfrak{h}^*} \mathrm{sdim}\, N_\nu e^\nu.$$

If $0 \to N_1 \to N \to N_2 \to 0$ is exact, then $\mathrm{sch}\, \mathrm{DS}_x(N) = sch\, \mathrm{DS}_x(N_1) + sch\, \mathrm{DS}_x(N_2)$.

1.2. **Choice of x.** In this paper we consider DS_x for $x \in \dot{\mathfrak{g}}$:

(1) $$x \in \dot{\mathfrak{g}}_{\bar{1}}, \ \ [x, x] = 0.$$

1.2.1. *Definition.* For $a \in \mathfrak{g}$ we denote by $\mathrm{supp}(a)$ the subset of $\Delta \cup \{0\}$ such that

$$a = \sum_{\beta \in \mathrm{supp}(a)} a_\beta,$$

where a_β is a non-zero vector in $\mathfrak{g}_\beta$.

1.2.2. *Definition.* We call $S \subset \dot{\Delta}_{\bar{1}}$ an *isotropic set* if S is a basis of an isotropic subspace in $\mathfrak{h}^*$.

Note that if $\mathrm{supp}(x)$ is an isotropic set, then x satisfies (1).

1.2.3. Let $\dot{G}$ be the Lie group of $\dot{\mathfrak{g}}_{\bar{0}}$. By [DS], Thm. 4.2 each x satisfying (1) is $\dot{G}$-conjugate to x', where $\mathrm{supp}(x')$ is an isotropic set; this gives a one-to-one correspondence between the $\dot{G}$-orbits for x satisfying (1) and $\dot{W}$-orbit of isotropic sets in $\dot{\Delta}$. In particular, for each x satisfying (1) there exists a base $\dot{\Sigma}$ such that x is $\dot{G}$-conjugate to x' such that $\mathrm{supp}(x')$ is an isotropic set and $\mathrm{supp}(x') \subset \dot{\Sigma}$. We call the cardinality of $\mathrm{supp}(x')$ the *rank* of x; 0 has the zero rank and the maximal rank is equal to the defect of $\dot{\mathfrak{g}}$.

Let $\mathfrak{t}$ be a Lie subalgebra of $\dot{\mathfrak{g}}_{\bar{0}}$ and N be a $\mathfrak{g}$-module which is $\mathfrak{t}$-finite (i.e., $\mathcal{U}(\mathfrak{t})v$ is finite-dimensional for each $v \in N$). Then the Lie group of $\mathfrak{t}$ acts on N. Moreover, any element g in this Lie group induces an isomorphism between the algebras $\mathfrak{g}_x$ and $\mathfrak{g}_{Ad_g(x)}$ and the corresponding modules $\mathrm{DS}_x(N)$ and $\mathrm{DS}_{Ad_g(x)}(N)$.

In particular, for a $\dot{\mathfrak{g}}$-integrable $\mathfrak{g}$-module N, this construction gives an isomorphism between $\mathrm{DS}_x(N)$ and $\mathrm{DS}_{x'}(N)$ with x' as above ($\mathrm{supp}(x')$ is an isotropic subset of a certain base $\dot{\Sigma}$).

1.2.4. Assume that $S := \mathrm{supp}(x)$ is an isotropic set.

It is shown in [DS], Lemma 6.3 that $\dot{\mathfrak{g}}_x$ a finite-dimensional Kac-Moody superalgebra with the roots

$$\dot{\Delta}_x := (S^{\perp} \cap \dot{\Delta}) \setminus (S \cup (-S));$$

$\dot{\mathfrak{g}}_x$ can be identified with a subalgebra of $\dot{\mathfrak{g}}$ generated by the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \dot{\Delta}_x$ and $\dot{\mathfrak{h}}_x \subsetneq \dot{\mathfrak{h}}^x = \{h \in \dot{\mathfrak{h}} |\ S(h) = 0\}$ such that

$$\dot{\mathfrak{h}}_x \oplus (\sum_{\beta \in S} \mathbb{C}h_\beta) = \dot{\mathfrak{h}}^x, \ \ [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \subset \dot{\mathfrak{h}}_x \ \forall \alpha \in \dot{\Delta}_x.$$

Moreover, $\dot{\mathfrak{h}}_x$ is a Cartan subalgebra of $\dot{\mathfrak{g}}_x$ and $\dot{\mathfrak{g}}^x = \dot{\mathfrak{g}}_x \oplus [x, \dot{\mathfrak{g}}]$. Note that $\mathfrak{h}_x^*$ is identified with a subspace in $\mathfrak{h}^*$ and $S^{\perp} = \mathbb{C}S \oplus \mathfrak{h}_x^*$.

If $\dot{\Delta}_x$ is not empty, then $\dot{\Delta}_x$ is the root system of the Lie superalgebra $\dot{\mathfrak{g}}_x$. One can choose a set of simple roots $\dot{\Sigma}_x$ such that $\Delta^+(\dot{\Sigma}_x) = \Delta^+ \cap \dot{\Delta}_x$.

Let r be the rank of x (i.e., $|S| = r$). If $\dot{\mathfrak{g}} = A(m|n), B(m|n)$ or $D(m|n)$, then $\dot{\mathfrak{g}}_x = A(m-r|n-r), B(m-r|n-r)$ or $D(m-r|n-r)$ respectively. If $\dot{\mathfrak{g}} = C(n)$, G_3 or F_4, then $r = 1$ and $\dot{\mathfrak{g}}_x$ is the Lie algebra of type C_{n-2}, A_1 and A_2 respectively. If $\dot{\mathfrak{g}} = D(2|1; a)$, then $r = 1$ and $\dot{\mathfrak{g}}_x = \mathbb{C}$. One has

$$\text{defect } \dot{\mathfrak{g}}_x = \text{defect } \dot{\mathfrak{g}} - \text{rank} x.$$

It is easy to show (see [GS]) that $\mathrm{DS}_x(\mathfrak{g}) = \mathfrak{g}_x$ is the affinization of $\dot{\mathfrak{g}}_x$; we identify this algebra with

$$\mathfrak{g}_x = \sum_{s=-\infty}^{\infty} (\dot{\mathfrak{g}}_x t^s) \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \mathfrak{h}_x := \dot{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d;$$

then $\Delta_x := \Delta(\mathfrak{g}_x)$ is the affinization of $\dot{\Delta}_x$. One has

$$\mathfrak{h}_x^* = \dot{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0 \subset \mathfrak{h}^*, \quad S^\perp = \mathfrak{h}_x^* \oplus \mathbb{C}S.$$

Set $\Delta_x^+ := \Delta^+(\Sigma) \cap \Delta_x$ and consider the corresponding triangular decomposition of $\mathfrak{g}_x$. We will describe the base Σ_x which corresponds to Δ_x^+ below.

If $\dot{\Delta}_x$ is empty, then $\dot{\mathfrak{g}}_x = 0$ or $\dot{\mathfrak{g}}_x = \mathfrak{gl}_1$. If $\dot{\mathfrak{g}}_x = 0$ (i.e., $\dot{\mathfrak{g}} = A(n|n), A(n+1|n), B(n|n), D(n|n), C(2)$), then $\mathfrak{g}_x = \mathbb{C}K \times \mathbb{C}d$. If $\dot{\mathfrak{g}}_x = \mathfrak{gl}_1$ (i.e., $\dot{\mathfrak{g}} = D(n+1|n)$ or $D(2|1,a)$), then $\mathfrak{g}_x = \mathfrak{gl}_1^{(1)}$, $\Delta^+(\mathfrak{g}_x) = \mathbb{Z}_{>0}\delta$ and $\Sigma_x = \{\delta\}$.

If $\dot{\Sigma}_x$ is connected, then $\Sigma_x := \dot{\Sigma}_x \cup \{\delta - \theta_x\}$, where θ_x is the maximal root in $\Delta^+(\dot{\Sigma}_x)$.

If $\dot{\Sigma}_x$ is not connected, then $\dot{\Delta}_x = D_2$ (i.e., $\dot{\mathfrak{g}} = D(n+2|n)$ with x of the maximal rank). In this case $\Delta_x = D_2^{(1)}$ is a union of two copies of $A_1^{(1)}$ with the same imaginary roots, that is $\Sigma_x := \dot{\Sigma}_x \cup \{\delta - \theta_x^i\}_{i=1}^2$, where $\dot{\Delta}_x^+ = \{\theta_x^1, \theta_x^2\}$.

1.3. **Casimir operator.** Take x as in (1). The bilinear form $(-,-)$ induces an invariant bilinear form $(-,-)_x$ on $\mathfrak{g}_x$. If N is an integrable $\mathfrak{g}$-module, then $\mathrm{DS}_x(N)$ is an integrable $\mathfrak{g}_x$-module.

Let Ω be the Casimir operator for $\mathfrak{g}$ which corresponds to the invariant bilinear form $(-,-)$, see [K1], Ch. II. Let $\dot{\Delta}_x \neq \emptyset$. By [GS], the image of Ω is the Casimir operator for $\mathfrak{g}_x$. This implies $||\rho||^2 = ||\rho_x||_x^2$ and

$$[\mathrm{DS}_x(L_{\mathfrak{g}}(\lambda)) : L_{\mathfrak{g}_x}(\lambda')] \neq 0 \implies (\lambda + 2\rho, \lambda) = (\lambda' + 2\rho_x, \lambda')_x, \tag{2}$$

where $\lambda' \in \mathfrak{h}_x^* = S^\perp/\mathbb{C}S$.

1.4. **Duality.** The duality in $\mathcal{O}$ is defined by an anti-automorphism σ of $\mathfrak{g}$ which stabilizes the elements of $\mathfrak{h}$. By above, $\mathfrak{g}_x, \mathfrak{g}_{\sigma(x)}$ are identified with a subalgebra of $\mathfrak{g}$ which is σ-stable (in particular, $\mathfrak{g}_x = \mathfrak{g}_{\sigma(x)}$). It si not hard to see that the map $\Psi : \mathrm{DS}_x(N^\#) \to (DS_{\sigma(x)}(N))^\#$ defined by $\Psi(f)(v) := f(v)$ is an isomorphism of $\mathfrak{g}_x$-modules if $N \in \mathcal{O}$.

2. Integrable vacuum modules

In this section $\dot{\mathfrak{g}}$ is a finite-dimensional Kac-Moody algebra and $x \in \dot{\mathfrak{g}}_{\bar{1}}$ is such that $supp(x)$ has a maximal rank, that is $\dot{\mathfrak{g}}_x$ has zero defect. Recall that $\mathfrak{g}_x$ is the affinization of $\dot{\mathfrak{g}}_x$.

2.1. **Vacuum modules.** If $\mathfrak{p}$ is a Kac-Moody superalgebra with a Cartan subalgebra $\mathfrak{t}$, we denote by $L_{\mathfrak{p}}(\lambda)$ a simple highest $\mathfrak{p}$-module with the highest weight $\lambda \in \mathfrak{t}^*$. For an affine Kac-Moody superalgebra $\mathfrak{p}$ we denote by $Vac_{\mathfrak{p}}^k$ the vacuum module of level k and by $|0\rangle$ the vacuum vector. For $\mathfrak{p} = \mathfrak{g}$ we write simply $L(\lambda)$, Vac^k. Note that $L(k\Lambda_0) = Vac_k$ is the simple quotient of Vac^k and so it does not depend on the choice of Σ; we call $L(k\Lambda_0)$ a *simple vacuum module*; if $L(k\Lambda_0)$ is integrable, we call it an *integrable vacuum module.*

2.1.1. The character of an integrable vacuum module is given by the Kac-Wakimoto character formula, see [GK]. From the proof it follows that an integrable vacuum module is a unique integrable quotient of Vac^k (since the proof uses only the fact that $L(k\Lambda_0)$ is a $\mathfrak{g}^{\#}$-integrable quotient of Vac^k, so all integrable quoteints have the same character and thus such quotient is unique).

Recall that $\dot{\mathfrak{g}}^{\#} \neq D_2$. Let θ be the highest root of $\dot{\Delta}^{\#}$ and $e \in \dot{\mathfrak{g}}^{\#}$ be the corresponding root vector. From [K1], Lem. 3.4 it follows that a quotient Vac^k/I is $\mathfrak{g}^{\#}$-integrable if and only if $k \in \mathbb{Z}_{\geq 0}$ and I contains $f_0^{k+1}|0\rangle$ for $f_0 := et^{-1}$. Therefore $L(k\Lambda_0)$ is integrable if and only if $k \in \mathbb{Z}_{\geq 0}$; in this case

$$L(k\Lambda_0) = Vac^k/I(k), \quad \text{where } I(k) \text{ is generated by } f_0^{k+1}|0\rangle.$$

Note that the vector $f_0^{k+1}|0\rangle$ is singular if $\delta - \theta \in \Sigma$.

2.1.2. *Remark.* Recall that $L(k\Lambda_0)$ does not depend on the choice of $\dot{\Sigma}$. Combining § 1.2 and § 5, we see that computing $\mathrm{DS}_x(L(k\Lambda_0))$ we can always assume that *supp*(x) is an isotropic set which lies in S satisfying (P1), (P2), (P3) in § 5.

2.2. Theorem. *Let $x \in \dot{\mathfrak{g}}_{\bar{1}}$ be such that $[x,x] = 0$ and supp(x) has a maximal rank.*

(i) If $\dot{\mathfrak{g}}_x = 0$ and $k \neq -h^\vee$, then $\mathrm{DS}_x(L(k\Lambda_0))$ is one-dimensional.

(ii) Assume that $\dot{\Sigma}$ contains $S := \mathrm{supp}(x)$ and the following inclusion holds

$$(\mathbb{Q}_{\geq 0}\Sigma \cap S^{\perp}) \subset (\mathbb{Q}S + \mathbb{Q}_{\geq 0}\Sigma_x). \tag{3}$$

If $L(\lambda)$ is integrable and $(\lambda, S) = 0$, then

$$\mathrm{DS}_x(L(\lambda)) \cong L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x}).$$

(iii) If $\dot{\mathfrak{g}}_x \neq \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$, one has

$$\mathrm{DS}_x(L(k\Lambda_0)) \cong L_{\mathfrak{g}_x}(k\Lambda_0).$$

2.3. Proof of Theorem 2.2. For (i), (iii) we set $\lambda := k\Lambda_0 \in \mathfrak{h}^*$ (so $L(\lambda) = L(k\Lambda_0)$) and $S := supp(x)$. Using Remark 2.1.2, we assume for (i), (iii) that $S, \dot{\Sigma}$ satisfies (P1), (P3) of § 5, i.e. $S \subset \dot{\Sigma}$ and (3) holds except for $\dot{\mathfrak{g}} = D(n+2|n), D(n+1|n)$.

We introduce

$$\lambda' := \lambda|_{\mathfrak{h}_x} \in \mathfrak{h}_x^*.$$

Since $(\lambda, S) = 0$ one has $\dim L(\lambda)_{\lambda-\nu} = \delta_{0,\nu}$ for $\nu \in \mathbb{Z}S$. Thus the singular vector in $L(\lambda)$ has a non-trivial image in $\mathrm{DS}_x(L(k\lambda))$ which is singular; moreover,

$$[\mathrm{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\lambda')] = 1. \tag{4}$$

For (i) $\mathfrak{g}_x = \mathbb{C}K \times \mathbb{C}d$. By § 1.3, the Casimir $\Omega_x = 2(K + h^\vee)d$ acts on $\mathrm{DS}_x(L(k\Lambda_0))$ by a scalar, so d acts on $\mathrm{DS}_x(L(k\Lambda_0))$ by a scalar. Now (i) follows from (4).

For (ii), (iii) assume that $[\mathrm{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\lambda' - \nu')] \neq 0$ for some $\nu' \in \mathfrak{h}_x^*$ with $\nu' \neq 0$. Since $\lambda' - \nu'$ is a weight of $\mathrm{DS}_x(L(\lambda))$, there exists $\nu \in \mathfrak{h}^*$ such that $\nu|_{\mathfrak{h}_x} = \nu'$, $(\nu, S) = 0$ and $L(\lambda)_{\lambda-\nu} \neq 0$. In particular,

$$\nu \in \mathbb{Z}_{\geq 0}\Sigma \cap S^\perp. \tag{5}$$

Let us prove (ii). Combining (5) and (3), $\nu \in \mathbb{Q}S + \mathbb{Q}_{\geq 0}\Sigma_x$, that is $\nu' \in \mathbb{Q}_{\geq 0}\Sigma_x$. If Δ_x is empty, we obtain $\nu' = 0$, a contradiction. Now we assume that Δ_x is not empty. By §1.3, $\mathrm{DS}_x(L(\lambda))$ is $\mathfrak{g}_x$-integrable, so $L_{\mathfrak{g}_x}(\lambda'), L_{\mathfrak{g}_x}(\lambda' - \nu')$ are integrable modules and $||\lambda' - \nu' + \rho_x||^2 = ||\lambda' + \rho_x||^2$, that is

$$(\lambda' - \nu' + \rho_x, \nu') + (\lambda' + \rho_x, \nu') = 0.$$

Since $\mathfrak{g}_x$ has zero defect, the integrability of $L_{\mathfrak{g}_x}(\lambda')$ and $L_{\mathfrak{g}_x}(\lambda' - \nu')$ gives $(\lambda', \nu'), (\lambda' - \nu', \nu') \geq 0$ and $(\nu', \rho_x) > 0$ (for $\nu' \neq 0$), a contradiction. This establishes (ii). Recall that for our choice of $(S, \dot{\Sigma})$, (3) holds except for $\dot{\mathfrak{g}} = D(n+2|n), D(n+1|n)$. Thus (ii) implies (iii).

It remains to verify (iii) for $\dot{\mathfrak{g}} = D(n+2|n)$. Consider the short exact sequence

$$0 \to I(k) \to Vac^k \to L(k\Lambda_0) \to 0,$$

where $I(k)$ is the maximal proper submodule of Vac^k. It is easy to see that $\mathrm{DS}_x(Vac^k) = Vac^k(\mathfrak{g}_x)$. Thus the corresponding long exact sequence is

$$0 \to E \to \mathrm{DS}_x(I(k)) \xrightarrow{\phi} Vac^k(\mathfrak{g}_x) \xrightarrow{\psi} \mathrm{DS}_x(L(k\Lambda_0)) \to \Pi(E) \to 0.$$

Since $\mathrm{DS}_x(L(k\Lambda_0))$ is $\mathfrak{g}_x$-integrable, the image of ψ is an integrable quotient of Vac^k, that is $L_{\mathfrak{g}_x}(k\Lambda_0)$. Since $L_{\mathfrak{g}_x}(k\Lambda_0 - \nu')$ is a subquotient of $\mathrm{DS}_x(L(k\Lambda_0))$ and $\nu' \neq 0$, it is a subquotient of $\Pi(E)$. Therefore $\Pi(L_{\mathfrak{g}_x}(k\Lambda_0 - \nu'))$ is a subquotient of $\mathrm{DS}_x(I(k))$. Take Σ such that $||\alpha_0||^2 > 0$. By § 2.1, $I(k)$ is generated by a singular vector of the weight $k\Lambda_0 - (k+1)\alpha_0$. Therefore

$$\lambda' - \nu' = k\Lambda_0 - (k+1)\alpha_0 - \mu',$$

where $\mu' = \mu|_{\mathfrak{g}_x}$ for some $\mu \in \mathfrak{h}^*$ such that $\mu \in \mathbb{Z}_{\geq 0}\Sigma$. Then

$$(\nu', \Lambda_0) \geq k + 1. \tag{6}$$

Take $S := \{\varepsilon_{i+1} - \delta_i\}_{i=1}^n$ and $\dot{\Sigma} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \ldots, \varepsilon_{n+1} - \delta_n, \delta_n \pm \varepsilon_{n+2}\}$; then $\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$, so $||\alpha_0||^2 = 2$. One has

$$\Sigma_x = \{\varepsilon_1 \pm \varepsilon_{n+2}; \delta - (\varepsilon_1 \pm \varepsilon_{n+2})\}, \quad \rho_x = 2\Lambda_0 + \varepsilon_1.$$

One readily sees that $(\mathbb{C}\Sigma \cap S^\perp) \subset (\mathbb{C}S + \mathbb{C}\Sigma_x)$, so $\nu' \in \mathbb{C}\Sigma_x$, so

$$\nu' = j\delta - s_+(\varepsilon_1 + \varepsilon_{n+2}) - s_-(\varepsilon_1 - \varepsilon_{n+2}).$$

The integrability of $L_{\mathfrak{g}_x}(k\Lambda_0-\nu')$ implies $0\le s_\pm\le k/2$. In addition, (2) gives

$$(k+2)j-s_+-s_-=s_+^2+s_-^2,$$

so $j\le k/2$. However, $j=(\nu',\Lambda_0)\ge k+1$ by (6). This contradiction completes the proof.

2.4. Example: $\mathfrak{g}=\mathfrak{sl}(1|2)^{(1)}, k=-1$. Take $\mathfrak{g}=\mathfrak{sl}(1|2)^{(1)}$ with $\Sigma=\{\delta-\varepsilon_1+\delta_2,\varepsilon_1-\delta_1,\delta_1-\delta_2\}$ and $S=\{\varepsilon_1-\delta_1\}$. Using the character formula (3.20) in [KW4] it is not hard to show that $\mathrm{DS}_x((L(-\Lambda_0))$ is not one-dimensional.

3. DS functor for vertex superalgebras

3.1. Vertex algebras. Recall that a vertex (super)algebra $V=V_{\bar 0}\oplus V_{\bar 1}$ is a vector superspace endowed with a vacuum vector $|0\rangle$, an even linear endomorphism T and a parity preserving linear map

$$Y:V\to(\operatorname{End}V)[[z,z^{-1}]],\quad a\mapsto Y(a,z)=\sum_{n\in\mathbb{Z}}a_{(n)}z^{-n-1}$$

subject to the following axioms $(a,b\in V, m,n\in\mathbb{Z})$

(translation covariance) $[T,Y(a,z)]=\partial_zY(a,z)$;

(vacuum) $T|0\rangle=0$; $Y(|0\rangle,z)=Id_V$; $a_{(-1)}|0\rangle=a$, $a_{(n)}|0\rangle=0$ for $n\ge 0$;

and the locality axiom which we use in the Borcherds form

$$(a_{(m)}b)_{(n)}=\sum_{i=0}^{\infty}(-1)^i\binom{m}{i}\big(a_{(m-i)}b_{(n+i)}-(-1)^{m+p(a)p(b)}b_{(m+n-i)}a_{(i)}\big).$$

For $m=0$ this gives

$$(a_{(0)}b)_{(n)}=[a_{(0)},b_{(n)}].\tag{7}$$

Note that T is "determined" by Y, i.e.

$$Ta=a_{(-2)}|0\rangle.\tag{8}$$

3.1.1. *Modules.* A weak module over a vertex superalgebra V in a vector superspace M with a parity preserving linear map

$$Y^M:V\to(EndM)[[z,z^{-1}]]\quad a\mapsto Y_M(a,z)=\sum_{n\mathbb{Z}}a^M_{(n)}z^{-n-1},$$

such that for each $v\in M$ one has $a^M_{(n)}v=0$ for $n>>0$, $Y_M(|0\rangle,z)=Id_M$ and $a^M_{(m)},b^M_{(n)}$ satisfy the locality axiom. As above, the locality axiom gives

$$(a_{(0)}b)^M_{(n)}=[a^M_{(0)},b^M_{(n)}].\tag{9}$$

An ideal of a vertex algebra is a subspace $I \subset V$ such that $a_{(m)}b, b_{(m)}a \in I$ for each $a \in I, b \in V, m \in \mathbb{Z}$. If I is an ideal of V, then the quotient V/I inherits the structure of a vertex algebra.

If I is an ideal of V, the V/I-modules are the V-modules annihilated by I, that is $a^M_{(m)} = 0$ for each $a \in I, m \in \mathbb{Z}$. We say that an ideal I is generated by a set E if I is a minimal ideal containing E. The locality axiom implies that in this case M is V/I-module if and only if $a^M_{(m)} = 0$ for each $a \in E, m \in \mathbb{Z}$.

3.2. Definition of $\mathrm{DS}_x(V)$. Let V be a vertex superalgebra and $x \in V$ be such that

(10) $$x \in V_{\bar{1}}, \quad x_{(0)}x = 0 \quad \text{and} \quad |0\rangle \notin Im x_{(0)}.$$

By (7) one has $x^2_{(0)} = 0$. We define the vector space $\mathrm{DS}_x(V)$ as follows:

$$\mathrm{DS}_x(V) := \mathrm{Ker}_V\, x_{(0)}/Im_V x_{(0)}.$$

By the vacuum axiom, $|0\rangle \in \mathrm{Ker}\, x_{(0)}$, so $|0\rangle$ has a non-zero image $|0\rangle' \in \mathrm{DS}_x(V)$. From the translation axiom $[T, x_{(0)}] = 0$, so T induces an even map $T' \in End\,\mathrm{DS}_x(V)$.

Take $b \in \mathrm{Ker}_V\, x_{(0)}$. From (7) it follows that $[b_{(n)}, x_{(0)}] = 0$ for each n, so $b_{(n)}$ induces $b'_{(n)} \in End(\mathrm{DS}_x(V))$ and $b'_{(n)} = 0$ if $b \in Im_V x_{(0)}$. This gives a parity preserving linear map

$$\mathrm{DS}_x(V) \to (End\,\mathrm{DS}_x(V))[[z, z^{-1}]] \quad b \mapsto Y'(b, z) = \sum_{n \in \mathbb{Z}} b'_{(n)} z^{-n-1}.$$

The space $\mathrm{DS}_x(V)$ equipped with $|0\rangle', T'$ and the fields $Y'(b, z)$ form a vertex algebra (the axioms for V' follow from the corresponding axioms for the vertex algebra V). We denote this vertex algebra by $\mathrm{DS}_x(V)$.

3.2.1. *Remark.* A vertex algebra V is $\mathbb{Z}_{\geq 0}$-graded if $V = \oplus_{s=0}^{\infty} V_s$ with

$$deg(a_{(j)}b) = deg(a) + deg(b) - j - 1,$$

where *deg* stands for the degree of a homogeneous vector in V. We claim that the condition $|0\rangle \notin Im a_{(0)}$ holds for each $a \in V$ if V is a $\mathbb{Z}_{\geq 0}$-graded vertex algebra with $V_0 = \mathbb{C}|0\rangle$.

Indeed, assume that $deg(a_{(0)}b) = 0$ for homogeneous a, b. Then $deg(a) + deg(b) = 1$, so a or b lie in $V_0 = \mathbb{C}|0\rangle$. However, $|0\rangle_{(0)} = 0$ and $a_{(0)}|0\rangle = 0$ for each a, that is $a_{(0)}b = 0$. Hence $|0\rangle \notin Im a_{(0)}$ for each $a \in V$.

3.2.2. *Modules.* Let M be a V-module. The condition $x_{(0)}x = 0$ gives $[x^M_{(0)}, x^M_{(0)}] = 0$. We introduce

$$\mathrm{DS}_x(M) = \mathrm{Ker}_M\, x^M_{(0)}/Im_M x^M_{(0)}.$$

Using (9) it is easy to check that $\mathrm{DS}_x(M)$ inherits a structure of $\mathrm{DS}_x(V)$-module (i.e., Y^M induces a map $\mathrm{DS}_x(V) \to (End\,\mathrm{DS}_x(M))[[z, z^{-1}]]$ which satisfy the corresponding axioms).

3.3. Affine vertex superalgebras. Let $\dot{\mathfrak{g}}$ be a finite-dimensional Kac-Moody superalgebra and let $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$.

By [FZ], the vacuum module $Vac^k(\mathfrak{g})$ has a structure of a vertex superalgebra with

$$Y(at^{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} (at^n) z^{-n-1} \quad \text{for } a \in \dot{\mathfrak{g}}. \tag{11}$$

We denote this vertex superalgebra by $V^k(\mathfrak{g})$.

The weak $V^k(\mathfrak{g})$-modules are the restricted $[\mathfrak{g}, \mathfrak{g}]$-module of level k (M is "restricted" if for each $v \in M$ one has $(\dot{\mathfrak{g}}t^s)v = 0$ for $s >> 0$).

The above correspondence between $V^k(\mathfrak{g})$-modules and $[\mathfrak{g}, \mathfrak{g}]$-modules implies that the maximal proper submodule $I(k)$ of $Vac^k(\mathfrak{g})$ is the maximal ideal in the vertex algebra $V^k(\mathfrak{g})$. Moreover, if $I(k)$ is generated by E as a $\mathfrak{g}$-module, then $I(k)$ is generated by E as an ideal in $V^k(\mathfrak{g})$. In particular, $L(k\Lambda_0)$ inherits a structure of a vertex superalgebra, which is simple; it is denoted by $V_k(\mathfrak{g})$.

For $\dot{\mathfrak{g}} = 0$ or $\dot{\mathfrak{g}} = \mathbb{C}$, the vacuum module $Vac^k(\mathfrak{g})$ is one-dimensional and $V^k(\mathfrak{g}) = V_k(\mathfrak{g})$ is a one-dimensional vertex algebra.

If $\mathfrak{g}$ is a Lie algebra, then for $k \in \mathbb{Z}_{\geq 0}$ the $V_k(\mathfrak{g})$-modules correspond to the restricted integrable $\mathfrak{g}$-modules of level k, see [DLM], Thm. 3.7; these modules are completely reducible and the irreducible modules are the integrable highest weight modules of level k (there are infinitely many such modules and $V_k(\mathfrak{g})$ is a *rational* vertex algebra). The following result was proven in [GS], Thm. 6.3.1 for $\dot{\mathfrak{g}}^{\#} \neq D_2$.

3.3.1. Theorem. *If $L(k\Lambda_0)$ are integrable, then $V_k(\mathfrak{g})$-modules are the restricted $[\mathfrak{g}, \mathfrak{g}]$-module of level k which are $[\mathfrak{g}^{\#}, \mathfrak{g}^{\#}]$-integrable.*

Proof. Let $I(k)$ be the maximal proper submodule of Vac^k, so $L(k\Lambda_0) = Vac^k/I(k)$.

For $\dot{\mathfrak{g}}^{\#} \neq D_2$ consider the natural embedding $Vac^k(\mathfrak{g}^{\#}) \subset Vac^k$ and denote by $I^{\#}$ the maximal proper submodule of $Vac^k(\mathfrak{g}^{\#})$.

If $\dot{\mathfrak{g}}^{\#} = D_2 = A_1 \times A_1$ consider the natural embeddings $Vac^k(A_1)', Vac^k(A_1)'' \subset Vac^k$ which correspond to two copies of A_1 in D_2; let I', I'' be the maximal proper submodules in $Vac^k(A_1)', Vac^k(A_1)''$ respectively. Set $I^{\#} := I' + I''$.

By § 2.1, the submodule $I(k)$ is generated by $I^{\#}$. By above, a restricted $[\mathfrak{g}, \mathfrak{g}]$-module N of level k is $V_k(\mathfrak{g})$-module if and only if it is annihilated by $a_{(m)}$ for each $a \in I^{\#}$, $m \in \mathbb{Z}$. Since $\dot{\mathfrak{g}}^{\#}$ is a Lie algebra, N is annihilated by $a_{(m)}$ for each $a \in I^{\#}$, $m \in \mathbb{Z}$ if and only if N is $\mathfrak{g}^{\#}$-integrable ([DLM], Thm. 3.7). □

3.4. DS_x for affine vertex algebras. Fix $x \in \dot{\mathfrak{g}}_{\bar{1}}$ satisfying $[x, x] = 0$. View

$$x' := xt^{-1}|0\rangle$$

as a vector in $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g})$ respectively. Note that $x'_{(0)} = x$.

One has $x'_{(0)}x' = x(xt^{-1}|0\rangle) = 0$. The vertex algebras $V^k(\mathfrak{g}), V_k(\mathfrak{g})$ are $\mathbb{Z}_{\geq 0}$-graded (the grading is given by the action of $-d \in \mathfrak{g}$) and the zero component is spanned by $|0\rangle$. Hence x' satisfies (10).

Consider the vertex algebras $\mathrm{DS}_{x'}(V^k(\mathfrak{g})), \mathrm{DS}_{x'}(V_k(\mathfrak{g}))$.

It is easy to see that $\mathrm{DS}_x(Vac^k(\mathfrak{g}))$ is canonically isomorphic to $Vac^k(\mathfrak{g}_x)$ as a $\mathfrak{g}_x$-module. Choose a vacuum vector $|0\rangle$ in $Vac^k(\mathfrak{g})$ and let the vacuum vector $|0\rangle'$ in $Vac^k(\mathfrak{g}_x)$ be the image of $|0\rangle$.

3.4.1. Theorem. *Let $\mathfrak{g}$ be an affine (non-twisted) Lie superalgebra and let $x \in \dot{\mathfrak{g}}_{\bar{1}}$ be such that $[x,x] = 0$; set $x' := xt^{-1}|0\rangle$.*

(i) The canonical isomorphism $\mathrm{DS}_x(Vac^k(\mathfrak{g})) \xrightarrow{\sim} Vac^k(\mathfrak{g}_x)$ induces a vertex algebra isomorphism $DS_{x'}(V^k(\mathfrak{g})) \xrightarrow{\sim} V^k(\mathfrak{g}_x)$.

(ii) If $\mathrm{DS}_x(L_{\mathfrak{g}}(k\Lambda_0)) \cong L_{\mathfrak{g}_x}(k\Lambda_0)$, then ι induces the vertex algebra isomorphism

$$DS_{x'}(V_k(\mathfrak{g})) \xrightarrow{\sim} V_k(\mathfrak{g}_x).$$

Proof. By above, ι is an isomorphism of $\mathfrak{g}_x$-modules and $\iota(|0\rangle) = |0\rangle'$. If $V^k(\mathfrak{g}_x)$ is one-dimensional, this implies (i) and (ii). Assume that $V^k(\mathfrak{g}_x)$ is not one-dimensional. Then $\Delta(\mathfrak{g}_x) \neq \emptyset$. Since ι is an isomorphism of $\mathfrak{g}_x$-modules,

$$\iota(at^{-1}|0\rangle) = (\mathrm{DS}_x(a)t^{-1})|0\rangle'$$

for each $a \in \dot{\mathfrak{g}}$ such that $[x,a] = 0$. By (11) we obtain

$$Y(\iota(v), z) = \iota(Y(v,z)) \tag{12}$$

for each $v = bt^{-1}|0\rangle$ with $b \in \dot{\mathfrak{g}}_x$.

Let V be a vertex algebra and E be a subspace of V. Denote by $\langle E\rangle$ the smallest subspace V' of V which contains E and such that $b_{(j)}v \in \langle E\rangle$ for each $b, v \in V'$ and $j \in \mathbb{Z}$. The locality axiom and (8) imply that if V admits two vertex algebra structures $(|0\rangle, T, Y)$ and $(|0\rangle, T', Y')$ such that $Y(v,z) = Y'(v,z)$ for each $v \in E$, then these structures coincide on $\langle E\rangle$ (i.e., $TV = T'v$ and $Y(v,z) = Y'(v,z)$ for each $v \in \langle E\rangle$).

Now let $E \subset V^k(\mathfrak{g}_x)$ (resp., $E \subset V^k(\mathfrak{g}_x)$) be the span of $|0\rangle$ and $bt^{-1}|0\rangle$ with $b \in \dot{\mathfrak{g}}_x$. Since $V^k(\mathfrak{g}_x)$ and $V_k(\mathfrak{g}_x)$ are generated by $|0\rangle$ as a $[\mathfrak{g}_x, \mathfrak{g}_x]$-modules, $V^k(\mathfrak{g}_x) = \langle E\rangle$ (resp., $V^k(\mathfrak{g}_x) = \langle E\rangle$). Thus ι is an isomorphism of the vertex algebras. □

Using Theorem 2.2 we obtain the

3.4.2. Corollary. *If k is a non-negative integer, x has a maximal rank and $\dot{\mathfrak{g}}$ differ from $D(n+1|n), D(2|1,a)$, then $DS_{x'}(V_k(\mathfrak{g}))$ and $V_k(\mathfrak{g}_x)$ are isomorphic as vertex algebras.*

3.4.3. Take x as above. Let M be a weak $V^k(\mathfrak{g}_x)$-module which we view as a restricted $\mathfrak{g}_x$-modules of level k. One readily sees that, as a $\mathfrak{g}_x$-module, the $DS_{x'}(V^k(\mathfrak{g}))$-module $DS_{x'}(M)$ is $\mathrm{DS}_x(M)$, so $DS_{x'}$ for $V^k(\mathfrak{g})$-modules correspond to DS_x for $[\mathfrak{g},\mathfrak{g}]$-modules. We will denote the functor $DS_{x'}$ by DS_x.

3.4.4. Corollary. *Let $L(k\Lambda_0)$ be integrable. For any $V_k(\mathfrak{g})$-module M the $V^k(\mathfrak{g}_x)$-module $\mathrm{DS}_x(M)$ is a $V_k(\mathfrak{g}_x)$-module.*

Proof. By Theorem 3.3.1, M is $[\mathfrak{g}^\#,\mathfrak{g}^\#]$-integrable. Note that $\mathfrak{g}^\#_x$ is the image of $\mathfrak{g}^\#\cap Ker_{\mathfrak{g}}x$ in $\mathfrak{g}_x = Ker_x\mathfrak{g}/Im_x\mathfrak{g}$. Therefore $\mathrm{DS}_x(M)$ is $[\mathfrak{g}^\#_x,\mathfrak{g}^\#_x]$-integrable, so $\mathrm{DS}_x(M)$ is a $V_k(\mathfrak{g}_x)$-module by Theorem 3.3.1. □

4. Principal admissible vacuum modules

In this section we define admissible weights for affine Lie superalgebras and prove Theorem 4.4.2.

4.1. Affine Lie algebra case. Let $\dot{\mathfrak{t}}$ be a finite-dimensional simple Lie algebra; let $\mathfrak{t} = \dot{\mathfrak{t}}^{(1)}$ be the corresponding affine Lie algebra with a Cartan subalgebra $\mathfrak{h}$. We denote by Δ_{re} the set of real roots of $\mathfrak{t}$.

4.1.1. For a non-critical weight $\lambda\in\mathfrak{h}^*$ the set of λ-integral real roots is defined as

$$\Delta_{re}(\lambda) = \{\alpha\in\Delta_{re}|\ \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}\}.$$

For our purposes we consider only λs where $\mathbb{C}\Delta_{re}(\lambda)=\mathbb{C}\Delta_{re}$. In this case $\Delta_{re}(\lambda)$ is the set of real roots of an affine Lie algebra algebra $\bar{\mathfrak{t}}$ with the same Cartan algebra $\mathfrak{h}$ and the triangular decomposition induced by the triangular decomposition of $\mathfrak{t}$, i.e.

$$\Delta_{re}(\lambda)^+ := \Delta_{re}(\lambda)\cap\Delta^+.$$

We denote by $\rho,\overline{\rho}$ the Weyl vectors of $\mathfrak{t},\bar{\mathfrak{t}}$ respectively. The character of $L_{\mathfrak{t}}(\lambda)$ and the character of the highest weight $\bar{\mathfrak{t}}$-module $L_{\bar{\mathfrak{t}}}(\lambda+\rho-\overline{\rho})$ are related by the following formula:

$$Re^\rho \operatorname{ch} L_{\mathfrak{t}}(\lambda) = \overline{R}e^{\overline{\rho}}\operatorname{ch} L_{\bar{\mathfrak{t}}}(\lambda+\rho-\overline{\rho}), \tag{13}$$

where $R,\overline{R}$ stand for the respective Weyl denominators (see [KT1],[KT2] and references there).

4.1.2. *Admissible weights.* A non-critical weight $\lambda \in \mathfrak{h}^*$ is called *admissible* if $\mathbb{C}\Delta_{re}(\lambda) = \mathbb{C}\Delta_{re}$ and $L_{\bar{\mathfrak{t}}}(\lambda + \rho - \overline{\rho})$ is an integrable $\bar{\mathfrak{t}}$-module.

If λ is admissible, then $chL_{\bar{\mathfrak{t}}}(\lambda+\rho-\overline{\rho})$ is given by the Weyl-Kac character formula and $\operatorname{ch} L(\lambda)$, suitably normalized, is a ratio of theta functions, which is a modular function, see [KW1],[KW2]. The admissible weights were classified in [KW2]. An admissible weight λ (and a module $L_{\mathfrak{t}}(\lambda)$) is called *principal admissible* if $\Delta_{re}(\lambda) \cong \Delta_{re}$, that is $\bar{\mathfrak{t}} \cong \mathfrak{t}$; the principal admissible weights were classified in [KW5].

4.1.3. *Principal admissible levels.* A level k is called *principal admissible* if $k\Lambda_0$ is principal admissible.

It is easy to see that k is principal admissible if and only if

$$k + h^\vee = \frac{p + h^\vee}{u}, \text{ where } p, u \in \mathbb{Z}_{\geq 0}, u > 0 \ \ (p + h^\vee, u) = (u, r^\vee) = 1,$$

where $r^\vee$ is the lacity of $\dot{\mathfrak{t}}$ (see the definition below in 4.2.1).

4.1.4. The following Adamović-Milas conjecture [AM] was proven by T. Arakawa in [A].

Theorem, Arakawa, 2014.

Let k be an admissible level for an affine Lie algebra $\mathfrak{t}$. The $V_k(\mathfrak{t})$-modules in the category $\mathcal{O}$ are completely reducible and the irreducible modules are $L_{\mathfrak{t}}(\lambda)$, where λ are the principal admissible weights of level k.

4.2. Admissibility for affine Lie superalgebras. Let $\mathfrak{g}$ be a (non-twisted) affine Lie superalgebra.

4.2.1. *Lacity.* Let $\dot{\mathfrak{g}} \neq D(2|1, a)$. We call $\alpha \in \Delta$ (resp., $\alpha \in \dot{\Delta}$) a short root if $|(\alpha, \alpha)|$ takes the smallest non-zero value. We define the lacity for $\mathfrak{g}$ and for $\dot{\mathfrak{g}}$ as

$$r^\vee = \frac{2}{|(\alpha, \alpha)|},$$

where α is a short root. Observe that the lacities for Δ and for $\dot{\Delta}$ are equal. Moreover, this lacity is equal to the lacity of $\dot{\mathfrak{g}}^\#$ if $\dot{\mathfrak{g}} \neq B(0|n)$; for $\dot{\mathfrak{g}} = B(0|n)$ one has $r^\vee = 4$.

The set $\Delta_{re}(\lambda)$ was introduced in [GK]. As for Lie algebra case, we define the admissible weights as follows.

4.2.2. *Definitions.* A non-critical weight $\lambda \in \mathfrak{h}^*$ is *admissible* if $\mathbb{C}\Delta_{re}(\lambda) = \mathbb{C}\Delta_{re}$ and $L_{\overline{\mathfrak{g}}}(\lambda + \rho - \overline{\rho})$ is an integrable $\overline{\mathfrak{g}}$-module.

An admissible weight λ is called *principal admissible* if $\Delta_{re}(\lambda) \cong \Delta_{re}$.

We say that k is an *admissible* (resp., *principal admissible*) level if $k\Lambda_0$ is admissible (resp., principal admissible).

By [GK], Thm. 11.2.3, $\mathrm{ch}\, L(k\Lambda_0)$ is given by (13) if k is admissible.

4.3. Principal admissible levels. It is not hard to show that for $\dot{\mathfrak{g}} \neq D(2|1,a)$ the level k is principal admissible if and only if

$$k + h^\vee = \frac{p + h^\vee}{u}, \text{ where } p, u \in \mathbb{Z}_{\geq 0}, u > 0 \quad (r^\vee(p + h^\vee), u) = 1,$$

where $r^\vee$ is the lacity of $\dot{\mathfrak{g}}$. Note that $r^\vee(p + h^\vee)$ is integral: $h^\vee$ is integral for $\dot{\mathfrak{g}} \neq B(m|n), m \leq n$, and $h^\vee = n - m + \frac{1}{2}$ for $\dot{\mathfrak{g}} = B(m|n), m \leq n$.

Let k be a principal admissible level. Then $\Delta_{re}(\overline{\mathfrak{g}}) = \dot{\Delta} + \mathbb{Z}u\delta$, where u is as above and the formula (13) takes the form

$$Re^\rho \,\mathrm{ch}\, L_{\mathfrak{g}}(k\Lambda_0) = \overline{R}e^{\overline{\rho}}\,\mathrm{ch}\, L_{\overline{\mathfrak{g}}}(p\overline{\Lambda}_0). \tag{14}$$

Note that $\Delta_{re}(\overline{\mathfrak{g}}) \cap \Delta^+$ vhas the base $\dot{\Sigma} \cup \{\alpha_0'\}$, where

$$\alpha_0' = (u-1)\delta + \alpha_0,$$

where $\Sigma = \dot{\Sigma} \cup \{\alpha_0\}$.

Recall that for $x \in \dot{\mathfrak{g}}$ such that $\dot{\Delta}_x$ is non-empty, $\dot{\mathfrak{g}}$ and $\mathrm{DS}_x(\dot{\mathfrak{g}})$ have the same dual Coxeter numbers. If, in addition, $\dot{\Delta}_x$ has rank more than one, then $\dot{\mathfrak{g}}$ and $\mathrm{DS}_x(\dot{\mathfrak{g}})$ have the same lacity $r^\vee$, so the principal admissible levels for $\mathfrak{g}$ and $\mathrm{DS}_x(\mathfrak{g})$ coincide. If $\dot{\Delta}_x$ has rank one, then the lacity of $\dot{\mathfrak{g}}$ is 1 for $A(n \pm 1|n)$ and 2 for other cases, whereas the lacity of $\mathrm{DS}_x(\dot{\mathfrak{g}})$ is 1; hence each principal admissible levels for $\mathfrak{g}$ is principle admissible for $\mathrm{DS}_x(\mathfrak{g})$.

4.4. Vacuum modules for principal admissible levels. Retain notation of § 4.3.

4.4.1. Take $x \in \dot{\mathfrak{g}}$ satisfying (1) such that x has a maximal rank, i.e.

$$\dot{\mathfrak{t}} := \mathrm{DS}_x(\dot{\mathfrak{g}})$$

has zero defect. We denote by $I_{\mathfrak{t}}(k)$ the maximal proper submodule of $Vac_{\mathfrak{t}}^k$. Let k be an admissible level for $\mathfrak{t}$. The vacuum module $Vac_{\mathfrak{t}}^k$ has a singular vector of weight $r_0'.k\Lambda_0$, where

$$r_0' := r_{\alpha_0'} \in W.$$

From [F] it follows that in the case when $\dot{\mathfrak{t}}$ is a simple Lie algebra, this singular vector generates $I_{\mathfrak{t}}(k)$.

4.4.2. Theorem. *Let $\dot{\mathfrak{g}} \neq B(n+1|n)$ be a finite-dimensional Kac-Moody algebra and let k be a principal admissible level. Let $x \in \dot{\mathfrak{g}}_{\bar{1}}$ be of the maximal rank.*

Assume that $\mathfrak{t} := \mathfrak{g}_x$ satisfies the following: $\dot{\mathfrak{t}}$ is simple,

(A1) $I_{\mathfrak{t}}(k)$ is generated by a singular vector of weight $r_0'.k\Lambda_0$;

(A2) any irreducible $V_k(\mathfrak{t})$-module in the category $\mathcal{O}$ is principal admissible.

Then

(i) $\mathrm{DS}_x(L(k\Lambda_0) \cong L_{\mathfrak{g}_x}(k\Lambda_0)$ as $\mathfrak{g}_x$-modules;

(ii) $\mathrm{DS}_x(V_k(\mathfrak{g})) \cong V_k(\mathfrak{g}_x)$ as vertex algebras;

(iii) for any $V_k(\mathfrak{g})$-module N, $\mathrm{DS}_x(N)$ is a $V_k(\mathfrak{g}_x)$-module;

(iv) if N is a $V_k(\mathfrak{g})$-module in $\mathcal{O}$, then $\mathrm{DS}_x(N)$ is either zero or the direct sum of principal admissible modules of level k.

4.5. Proof of Theorem 4.4.2. Note that $\dot{\mathfrak{g}} \neq A(m|n), B(m|n), D(m|n)$ with $m = n, n+1$ and $D(n+2|n)$. Using Remark 2.1.2, we assume for that $S, \dot{\Sigma}$ satisfies (P1), (P2), (P3) of § 5, i.e.

$$S \subset \dot{\Sigma}, \quad (S, \theta) = 0, \quad ||\theta||^2 = 2,$$

where θ is the maximal root in $\Delta^+(\dot{\Sigma})$, and (20) holds.

In particular, $\alpha_0 := \delta - \theta$ is the affine root for $\mathfrak{g}$ and for $\mathfrak{t}$.

Since $L(k\Lambda_0)$ is $\dot{\mathfrak{g}}$-integrable, we can (and will) assume that $supp(x) = S$.

We fix the $\mathbb{Z}_2$-grading on Vac^k and all its subquotients by letting the highest weight vector to be even. For a $\mathbb{Z}_2$-graded space E we write $\dim E = (a|b)$ if $\dim E_{\bar{0}} = a, \dim E_{\bar{1}} = b$. Retain notation of § 4.3.

4.5.1. Denote by $I(k)$ the maximal submodule of Vac^k and by $I_{\overline{\mathfrak{g}}}(p)$ the maximal submodule of the vacuum $\overline{\mathfrak{g}}$-module $Vac^p_{\overline{\mathfrak{g}}}$. One has

$$Re^{-k\Lambda_0} chI(k) = Re^{-k\Lambda_0}(chVac^k - chL(k\Lambda_0)) = \dot{R} - Re^{-k\Lambda_0} chL(k\Lambda_0),$$

where $R, \dot{R}, \overline{R}$ are the Weyl denominators for $\Delta^+, \dot{\Delta}^+, \overline{\Delta}^+$ respectively; recall that $\dot{\Delta} \subset \overline{\Delta}$, so $\dot{\overline{R}} = \dot{R}$.

From (14) we have $Re^{-k\Lambda_0} chL(k\Lambda_0) = \overline{R}e^{-p\overline{\Lambda}_0} chL_{\overline{\mathfrak{g}}}(p\overline{\Lambda}_0)$. This gives

$$Re^{-k\Lambda_0} chI(k) = \dot{R} - \overline{R}e^{-p\overline{\Lambda}_0} chL_{\overline{\mathfrak{g}}}(p\overline{\Lambda}_0) = \overline{R}e^{-p\overline{\Lambda}_0} chI_{\overline{\mathfrak{g}}}(p). \tag{15}$$

By § 2.1, $I_{\overline{\mathfrak{g}}}(p)$ is generated by a singular vector v' of the weight

$$\mu := p\overline{\Lambda}_0 - (p+1)\alpha_0'.$$

Now the formula (15) can be rewritten as

$$Re^{-r_0'.k\Lambda_0}chI(k)=\overline{R}e^{-\mu}chI_{\overline{\mathfrak{g}}}(p) \tag{16}$$

since $k\dot{\Lambda}_0+\mu-p\overline{\Lambda}_0=k\Lambda_0-(p+1)\alpha_0'=r_0'.(k\Lambda_0)$.

Recall that $v'=f_0^{p+1}|0\rangle$, where $f_0\in\mathfrak{g}_{-\alpha_0'}$. For any $\beta\in S$ we have $(\alpha_0',\beta)=0$, so $[\mathfrak{g}_{-\alpha_0'},\mathfrak{g}_{\pm\beta}]=0$. Therefore $\mathfrak{g}_{\pm\beta}v'=0$ and so

$$\dim I_{\overline{\mathfrak{g}}}(p\Lambda_0)_\mu=(1|0),\quad I_{\overline{\mathfrak{g}}}(p\Lambda_0)_{\mu-\gamma}=0\ \text{ for }\gamma\in\mathbb{Z}S\setminus\{0\}. \tag{17}$$

4.5.2. Set

$$\mathcal{R}:=\{\sum_{\nu\in\mathbb{Z}_{\geq 0}\Sigma}a_\nu e^{-\nu}|\ a_\nu\in\mathbb{C}\},\quad P_S(\sum_{\nu\in\mathbb{Z}_{\geq 0}\Sigma}a_\nu e^{-\nu}):=\sum_{\nu\in\mathbb{Z}_{\geq 0}S}a_\nu e^{-\nu}.$$

Clearly, $\mathcal{R}$ has a ring structure; this ring does not have zero divisors. Note that P_S is a ring homomorphism (since $S\subset\Sigma$) and $P_S^2=P_S$.

The ring $\mathcal{R}$ contains $R,\dot{R},\overline{R},R^{-1},\overline{R}^{-1}$ and

$$P_S(R)=P_S(\dot{R})=P_S(\overline{R}).$$

Since $I_{\overline{\mathfrak{g}}}(p)$ is generated by a singular vector of weight μ, one has $e^{-\mu}chI_{\overline{\mathfrak{g}}}(p)\in\mathcal{R}$. By (17), $P_S(e^{-\mu}ch_{\overline{\mathfrak{g}}}I(p))=1$. Using (16) we get

$$e^{-r_0'.k\Lambda_0}chI(k)\in\mathcal{R},\quad P_S(e^{-r_0'.k\Lambda_0}chI(k))=1, \tag{18}$$

By (16), $r_0'.k\Lambda_0$ is the highest weight of $I(k)$ and $\dim I(k)_{r_0'.k\Lambda_0}=1$. Thus $I(k)_{r_0'.k\Lambda_0}$ is spanned by an even singular vector. By (18), this vector has non-zero image in $\mathrm{DS}_x(I(k))$ $\mathrm{DS}_x(I(k))_{r_0'.k\Lambda_0}$ is spanned by this image.

We conclude that $\mathrm{DS}_x(I(k))_{r_0'.k\Lambda_0}$ is spanned by an even singular vector, which we denote by v_0.

4.5.3. Recall that $\mathrm{DS}_x(Vac^k)=Vac_{\mathfrak{t}}^k$. Consider the short exact sequence

$$0\to I(k)\to Vac^k\to Vac_k\to 0$$

and the corresponding long exact sequence

$$0\to E\to\mathrm{DS}_x(I(k))\xrightarrow{\phi}Vac_{\mathfrak{t}}^k\xrightarrow{\psi}\mathrm{DS}_x(Vac_k)\to\Pi(E)\to 0.$$

By (A1), $I_{\mathfrak{t}}(k)$ is generated by a singular vector v_0' of weight $r_0'.k\Lambda_0$. Since v_0,v_0' are singular, $\phi(v_0)$ is proportional to v_0'. There are two possibilities: either $\phi(v_0)=v_0'$ (up to a non-zero scalar) or $\phi(v_0)=0$.

Assume that $\phi(v_0) = 0$. Since v_0 spans $\mathrm{DS}_x(I(k))_{r_0'.k\Lambda_0}$ one has $v_0' \notin Im\phi = Ker\psi$. Since $v_0 \in Ker\phi$ one has

$$\dim E_{r_0'.k\Lambda_0} = \dim \mathrm{DS}_x(I(k))_{r_0'.k\Lambda_0} = (1|0). \tag{19}$$

Since $v_0' \notin Ker\psi$, the $\mathfrak{t}$-module $\mathrm{DS}_x(Vac_k)$ has an even indecomposable subquotient of length two with the socle $L_{\mathfrak{t}}(r_0'.k\Lambda_0)$ and the cosocle $L_{\mathfrak{t}}(k\Lambda_0)$. Since $Vac_k \cong L(k\Lambda_0)$ is self-dual, $\mathrm{DS}_x(Vac_k)$ is also self-dual (see § 1.4); thus $\mathrm{DS}_x(Vac_k)$ has an even indecomposable subquotient of length two with the cosocle $L_{\mathfrak{t}}(r_0'.k\Lambda_0)$ and the socle $L_{\mathfrak{t}}(k\Lambda_0)$. Since $I_{\mathfrak{t}}(k)$ is generated by a singular vector of weight $r_0'.k\Lambda_0$, one has $[Vac_{\mathfrak{t}}^k : L_{\mathfrak{t}}(r_0'.k\Lambda_0)] = 1$, so $Im\psi$ does not have such subquotient. Then $\Pi(E)$ has an even subquotient $L_{\mathfrak{t}}(r_0'.k\Lambda_0)$, which contradicts to (19).

We conclude that $\phi(v_0) = v_0'$ up to a non-zero scalar. Denote by a a preimage of v_0 in $I(k) \subset Vac^k$. Let N be a $V_k(\mathfrak{g})$-module and $\mathrm{DS}_x(N) \neq 0$. Since $Vac_k = Vac^k/I(k)$, § 3.1.1 gives $Y(a,z)N = 0$, so $Y(v_0,z)\,\mathrm{DS}_x(N) = 0$. Since $Vac_{\mathfrak{t},k} = Vac_{\mathfrak{t}}^k/I_{\mathfrak{t}}(k)$ with $I_{\mathfrak{t}}(k)$ generated by v_0', § 3.1.1 implies that $\mathrm{DS}_x(N)$ is a $V_k(\mathfrak{t})$-module. This establishes (iii).

Let us prove that $\mathrm{DS}_x(Vac_k) = L_{\mathfrak{t}}(k\Lambda_0)$. Clearly, $[\mathrm{DS}_x(Vac_k) : L_{\mathfrak{t}}(k\Lambda_0)] = 1$. Let $L_{\mathfrak{t}}(\lambda'')$ be a subquotient of $\mathrm{DS}_x(Vac_k)$ and $\lambda'' \neq k\Lambda_0$. By (iii) and (A2), λ'' is a $\mathfrak{t}$-admissible weight. One has $\lambda'' = k\Lambda_0 - (\nu|_{\mathfrak{h}_S})$ for some $\nu \in (\mathbb{Z}_{\geq 0}\Sigma \cap S^{\perp})$. Recall that $(S, \dot{\Sigma})$ satisfies (20), so $\nu|_{\mathfrak{h}_S} \in \mathbb{Z}\Sigma_S$, which contradicts to Lemma 4.7. This gives (i); (ii) follows from Theorem 3.4.1 (ii). □

4.6. Corollary. *Let $\dot{\mathfrak{g}}$ is one of the following algebras: $A(m|n), C(n)$; $B(m|n), m \geq n+2$; $D(m|n), m \neq n+1, n+2$, $B(n|n), F(4)$ or $G(2)$. Take $x \in \dot{\mathfrak{g}}_{\overline{1}}$ such that supp(x) is maximal. Let k be an admissible level. Then*

(i) $\mathrm{DS}_x(L(k\Lambda_0) \cong L_{\mathfrak{g}_x}(k\Lambda_0)$ as $\mathfrak{g}_x$-modules;

(ii) $\mathrm{DS}_x(V_k(\mathfrak{g})) \cong V_k(\mathfrak{g}_x)$ as vertex algebras;

(iii) for any $V_k(\mathfrak{g})$-module N, $\mathrm{DS}_x(N)$ is a $V_k(\mathfrak{g}_x)$-module;

(iv) if N is a $V_k(\mathfrak{g})$-module in $\mathcal{O}$, then $\mathrm{DS}_x(N)$ is either zero or the direct sum of principal modules of level k.

Proof. For $\dot{\mathfrak{g}}_x \neq 0$, the assumption (A1) of Theorem 4.4.2 follows from [F] and the assumption (A2) follows from Theorem 4.1.4. This gives (i)–(iii) for $\dot{\mathfrak{g}}_x \neq 0$; (iv) follows from (iii) and Theorem 4.1.4.

If $\dot{\mathfrak{g}}_x = 0$, then (i) is a particular case of Theorem 2.2 (i); moreover, (ii)-(iv) follow from (i). □

4.7. The following lemma was used in the proof.

Lemma. *Let $\mathfrak{t}$ has zero defect and let k be a principal admissible level. If λ is an admissible weight such that $k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$, then $\lambda = k\Lambda_0$.*

Proof. Since $k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$ one has $\Delta_{re}(\lambda) = \Delta_{re}(k\Lambda_0)$. Set

$$\lambda' := \lambda + (p + h^\vee)(1 - \frac{1}{u})\Lambda_0.$$

One readily sees that λ' is a dominant weight of level p. One has $p\Lambda_0 - \lambda' = k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$. Since λ' is dominant,

$$0 \leq (\Lambda_0, p\Lambda_0 - \lambda') = -(\lambda', \Lambda_0) \quad \text{and} \quad 0 \leq (p\Lambda_0 - \lambda', \lambda') = p(\Lambda_0, \lambda') - (\lambda', \lambda').$$

Therefore $(\lambda', \lambda') = (\Lambda_0, \lambda') = 0$. Since $p\Lambda_0 - \lambda' = k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$, we obtain $k\Lambda_0 - \lambda = 0$ as required. □

5. Appendix

We fix the standard triangular decomposition in $\dot{\Delta}_{\bar{0}}$ and consider the bases $\dot{\Sigma}$ which are compatible with this decomposition. For each base $\dot{\Sigma}$ denote by $\theta_{\dot{\Sigma}}$ the maximal root of $\Delta^+(\dot{\Sigma})$.

Let $\mathcal{S}$ be the set of maximal isotropic subsets of $\Delta_{\bar{1}}$. Consider the action of the Weyl group $\dot{W}$ on $\mathcal{S}$. For each orbit it is not hard to give an example of a pair $(S, \dot{\Sigma})$ such that S is a representative of this orbit and

(P1) $S \subset \dot{\Sigma}$;

(P2) $\theta_{\dot{\Sigma}} \in \dot{\Delta}^{\#}$ and $(\theta_{\dot{\Sigma}}, S) = 0$ for $\dot{\mathfrak{g}} \neq A(m|n), B(m|n), D(m|n)$ with $m = n, n+1$;

(P3) if $\dot{\mathfrak{g}} \neq D(n+1, n), D(n+2|n)$, then the following inclusion holds

$$(\mathbb{Q}_{\geq 0}\Sigma \cap S^\perp) \subset (\mathbb{Q}S + \mathbb{Q}_{\geq 0}\Sigma_S), \tag{20}$$

where $\Sigma = \{\delta - \theta_{\dot{\Sigma}}\} \cup \dot{\Sigma}$ is a base for $\Delta = \dot{\Delta}^{(1)}$.

For instance, for $B(m,n), D(m,n), n > m$ one has $\dot{\Delta}^{\#} = C_n$. We take $S := \{\varepsilon_i - \delta_{i+1}\}_{i=1}^m$ and

$$\dot{\Sigma} := \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \ldots, \varepsilon_m - \delta_{m+1}, \delta_{m+1} - \delta_{m+2}, \ldots, \delta_{n-1} - \delta_n, a\delta_n\},$$

where $a = 1$ for $B(m|n)$ and $a = 2$ for $D(m|n)$. One has $\theta = 2\delta_1$, so (P1), (P2) are satisfied. One has

$$\dot{\Sigma}_S = \{\delta_1 - \delta_{m+2}, \ldots, \delta_{n-1} - \delta_n, a\delta_n\},$$

of type $B(0|n-m)$ for $B(m|n)$ and C_{n-m} for $D(m|n)$; it is easy to see that (20) holds.

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